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Gauged Nonlinear Sigma Model in Light-Front Frame: Hamiltonian and BRST Formulations

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A gauged nonlinear sigma model in one-space one-time dimension is considered in the light-front frame. The theory is seen to possess a local vector gauge symmetry. The light-front Hamiltonian and BRST formulations of this theory are investigated under some specific light-cone gauges.

1. INTRODUCTION

The O (N) nonlinear sigma models (NLSM) in one-space one-time ($(1 + 1)-$) dimension (Callen *et al.*, 1969; Candelas *et al.*, 1985; Coleman *et al.*, 1969; Henneaux and Mezincescu, 1985; Kulshreshtha *et al.*, 1993a; Maharana, 1983a,b; Mitra and Rajaraman, 1990a,b; Ruehl, 1991a,b, 1993, 1995, 1996; Zamolodchikov and Zamolodchikov, 1979), where the field sigma is a real *N*-component field, provide a laboratory for the various nonperturbative techniques for example, 1/*N*-expansion (Ruehl, 1991a,b, 1993, 1995, 1996), operator product expansion, and the low energy theorems (Callen *et al.*, 1969; Coleman *et al.*, 1969). These models are characterized by features like the renormalization and asymptotic freedom common to that of quantum chromodynamics, and they exhibit a nonperturbative particle spectrum, have no intrinsic scale parameter, possess the topological charges, and are very crucial in the context of conformal (Ruehl, 1991a,b, 1993, 1995, 1996) and string-field theories (Candelas *et al.*, 1985; Henneaux and Mezincescu, 1985), where they appear in the classical limit (Callen *et al.*, 1969; Coleman *et al.*, 1969).

The Hamiltonian formulation of the gauge-non-invariant (GNI), O (N)-NLSM in $(1 + 1)$-dimension, has been studied in Maharana (1983a) and its two gauge-invariant (GI) versions have been constructed in Kulshreshtha *et al.* (1993a), where the Hamiltonian (Dirac, 1950, 1964) and Becchi–Rouet–Stora–Tyutini (BRST) (Becchi *et al.*, 1974; Henneaux and Teitelboim, 1992; Kulshreshtha, 1998;

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Kulshreshtha and Kulshreshtha, 1998; Kulshreshta *et al.*, 1993b, 1994a,b,c, 1995, 1999; Nemeschansky *et al.*, 1988; Tyutin, 1975) quantization of these GI models has also been studied in detail (Kulshreshtha *et al.*, 1993a). The NLSM studied in Kulshreshtha *et al.* (1993a); Maharana (1983a,b); and Mitra and Rajaraman (1990a,b) do not have any gauge fields in the theory. Corresponding to these models (Kulshreshtha *et al.*, 1993a; Maharana, 1983a,b; Mitra and Rajaraman, 1990a,b); if we consider models involving the gauge field, as proposed in the present paper, we obtain the so-called gauged-NLSM (GNLSM). In the present paper, we propose to study such a GNLSM obtained by gauging the usual NLSM (without involving the vector gauge field $A^{\mu}(x, t)$) (Kulshreshtha *et al.*, 1993a; Maharana, 1983a,b; Mitra and Rajaraman, 1990a,b) and investigate its canonical structure, constrained dynamics, and Hamiltonian (Dirac, 1950, 1964) and BRST (Becchi *et al.*, 1974; Henneaux and Teitelboim, 1992; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshta *et al.*, 1993b, 1994a,b,c, 1995, 1999; Nemeschansky *et al.*, 1988; Tyutin, 1975) formulations in the light-front (LF) frame on the hyperplanes $x^{+} = (x^{0} + x^{1})/\sqrt{2} =$ constant (Dirac, 1949). The Hamiltonian and BRST formulations of this GNLSM in the usual instant form (IF) of dynamics (on the hyperplanes $x^{0} =$ constant) (Dirac, 1949) has been investigated (Kulshreshtha, 2001).

The IF theory (Kulshreshtha, 2001) is seen to possess a set of five first-class constraints (where two constraints are primary and three are secondary) implying that the theory is a GI theory.

The LF theory under the present investigation is also seen to possess a set of five first-class constraints, however—now having three primary constraints and two secondary constraints—implying again that the theory under consideration is a GI theory. The LF Hamiltonian formulation of this GNLSM is investigated in the present paper under some specific light-cone (LC) gauges.

However, in the usual Hamiltonian formulation of a GI theory under some gauge-fixing conditions, one necessarily destroys the gauge invariance of the theory by fixing the gauge (which converts a set of first-class constraints into a set of second-class constraints, implying a breaking of gauge invariance under the gauge fixing). To achieve the quantization of a GI theory such that the gauge invariance of the theory is maintained even under the gauge fixing, one goes to a more generalized procedure called the BRST formulation (Becchi *et al.*, 1974; Henneaux and Teitelboim, 1992; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a,b, 1994a,b,c, 1995, 1999; Nemeschansky *et al.*, 1988; Tyutin, 1975). In the BRST formulation of a GI theory, the theory is rewritten as a quantum system that possesses a generalized gauge invariance called the BRST symmetry. For this, one enlarges the Hilbert space of the GI theory and replaces the notion of the gauge transformation, which shifts operators by c-number functions by a BRST transformation that mixes the operators having different statistics. In view of this, one introduces new anticommuting variables c

and $\bar{c}$ called the Faddeev–Popov ghost and antighost fields, which are Grassmann numbers on the classical level and operators in the quantized theory, and a commuting variable b called the Nakanishi–Lautrup field (Becchi *et al.*, 1974; Henneaux and Teitelboim, 1992; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshta *et al.*, 1993b, 1994a,b,c, 1995, 1999; Mitra and Rajaraman, 1990a,b; Nemeschansky *et al.*, 1988; Tyutini, 1975).

In the BRST formulation of a theory one thus embeds a GI theory into a BRST-invariant system, and the quantum Hamiltonian of the system (which includes the gauge-fixing contribution) commutes with the BRST charge operator Q as well as with the anti-BRST charge operator $\bar{Q}$. The new symmetry of the system (the BRST symmetry) that replaces the gauge invariance is maintained (even under gauge fixing) and hence projecting any state onto the sector of BRST and anti-BRST invariant states yields a theory that is isomorphic to the original GI theory. The unitarity and consistency of the BRST-invariant theory described by the gauge-fixed quantum Lagrangian is guaranteed by the conservation and nilpotency of the BRST charge Q.

In the next section, we briefly recapitulate the basics of the usual O (N)-NLSM (without gauge fields) (Kulshreshtha *et al.*, 1993a; Maharana, 1983a,b; Mitra and Rajaraman, 1990a,b) as well as that of the GNLSM in the instant form of dynamics (Kulshreshtha, 2001). In Section 3, we study the Hamiltonian formulation of the proposed GNLSM in the LF frame, and in Section 4, its BRST formulation under some specific light-cone gauges. The summary and discussions are finally given in Section 5.

2. A RECAPITULATION OF THE NONLINEAR SIGMA MODEL (NLSM) IN THE INSTANT FORM (IF)

2.1. The Usual (Ungauged) Theory

The O (N)-nonlinear sigma model in one-space one-time dimension in the usual IF (i.e., on the hyperplanes x^0 = constant) is described by the action (Callen *et al.*, 1969; Candelas *et al.*, 1985; Coleman *et al.*, 1969; Henneaux and Mezincescu, 1985; Kulshreshtha *et al.*, 1993a; Maharana, 1983a,b; Mitra and Rajaraman, 1990a,b; Ruehl, 1991a,b, 1993, 1995, 1996; Zamolodchikov and Zamolodchikov, 1979)

$$S = \int \mathscr{L}^N dx\, dt \tag{2.1a}$$

$$\mathscr{L}^N = \left[\frac{1}{2}\partial_\mu \sigma_k \partial^\mu \sigma_k + \lambda\left(\sigma_k^2 - 1\right)\right]; \quad k = 1, 2, \ldots, N \tag{2.1b}$$

$$\mathscr{L}^N = \left[\frac{1}{2}\left(\mathring{\sigma}_k^2 - \sigma_k'^2\right) + \lambda\left(\sigma_k^2 - 1\right)\right]; \quad k = 1, 2, \ldots, N \tag{2.1c}$$

here $\vec{\sigma} \equiv [\sigma_k(x,t); k = 1, 2, \ldots, N]$ is a multiplet of N real scalar fields in $(1+1)$-dimension and $\lambda(x,t)$ is another scalar field. The overdots and primes denote the time and space derivatives respectively. The field $\vec{\sigma}(x,t)$ maps the two-dimensional space-time into the N-dimensional internal manifold whose co-ordinates are $\sigma_k(x,t)$. This model is seen to possess a set of four second-class constraints (Kulshreshtha *et al.*, 1993a; Maharana, 1983a,b; Mitra and Rajaraman, 1990a,b):

$$\rho_1 = p_\lambda \approx 0 \tag{2.2a}$$

$$\rho_2 = \left[\sigma_k^2 - 1\right] \approx 0 \tag{2.2b}$$

$$\rho_3 = 2\sigma_k \, \Pi_k \approx 0 \tag{2.2c}$$

$$\rho_4 = \left(2\Pi_k^2 + 4\lambda \, \sigma_k^2 + 2\sigma_k \, \sigma_k''\right) \approx 0 \tag{2.2d}$$

where ρ_1 is a primary constraint and ρ_2, ρ_3, and ρ_4 are secondary constraints. Here Π_k and p_λ are the momenta canonically conjugate respectively to σ_k and λ. The nonvanishing equal-time Dirac brackets (DBs) of the theory are given by Kulshreshtha *et al.* (1993a) and Maharana (1983a)

$$\{\Pi_\ell(x), \Pi_m(y)\}_{\mathrm{D}} = \frac{-1}{\sigma_k^2}\,[\sigma_\ell(x)\Pi_m(y) - \Pi_\ell(x)\sigma_m(y)]\,\delta(x-y) \tag{2.3a}$$

$$\{\sigma_\ell(x), \Pi_m(y)\}_{\mathrm{D}} = \left[\delta_{\ell m} - \frac{\sigma_\ell(x)\sigma_m(y)}{\sigma_k^2}\right]\delta(x-y). \tag{2.3b}$$

For achieving the canonical quantization of the theory, one encounters the problem of operator ordering while going from DBs to the commutation relations. This problem could, however, be resolved as explained in Kulshreshtha *et al.* (1993a) and Maharana (1983a,b) by demanding that all the fields and field momenta after quantization become Hermitian operators and that all the canonical commutation relations be consistent with the hermiticity of these operators (Kulshreshtha *et al.*, 1993; Maharana, 1983a,b).

2.2. The Gauged Nonlinear Sigma Model (GNLSM)

In one of our earlier papers (Kulshreshtha, 2001), we have studied the GNLSM in the instant form (IF) of dynamics on the hyperplanes $x^0 =$ constant. This IF-GNLSM is described by the action in $(1+1)$-dimension (Kulshreshtha, 2001).

$$S = \int \mathscr{L} \, dx \, dt \tag{2.4a}$$

$$\mathscr{L} := \left[\frac{1}{2}\partial_\mu\sigma_k\partial^\mu\sigma_k + \lambda\left(\sigma_k^2 - 1\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu\partial^\mu\sigma_k \right.$$
$$\left. + \frac{1}{2}\, e^2 A_\mu A^\mu\right] \tag{2.4b}$$

$$\mathscr{L} := \left[\frac{1}{2}\left(\mathring{\sigma}_k^2 - \sigma_k'^2\right) + \lambda\left(\sigma_k^2 - 1\right) + \frac{1}{2}(\mathring{A}_1 - A_0')^2 - e(A_0\mathring{\sigma}_k - A_1\sigma_k') \right.$$
$$\left. + \frac{1}{2}e^2\left(A_0^2 - A_1^2\right)\right] \tag{2.4c}$$

$$F^{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu); \qquad g^{\mu\nu} := \mathrm{diag}(+1, -1). \tag{2.4d}$$

In the previous equation, the first term corresponds to a massless boson (which is equivalent to a massless fermion), the second term is the usual term involving the nonlinear constraint ($\sigma_k^2 - 1 \approx 0$) and the auxiliary field λ, the third term is the kinetic energy term of the electromagnetic vector-gauge field $A_\mu(x, t)$, the fourth term represents the coupling of the sigma field to the electromagnetic field, and the last term is the mass term for the vector gauge boson $A_\mu(x, t)$ and contains the signature of regularization.

This theory is seen to possess a set of five constraints (Kulshreshtha, 2001):

$$\Omega_1 = \Pi_0 \approx 0 \tag{2.5a}$$
$$\Omega_2 = p_\lambda \approx 0 \tag{2.5b}$$
$$\Omega_3 = [E' - e\Pi_k] \approx 0 \tag{2.5c}$$
$$\Omega_4 = \left[\sigma_k^2 - 1\right] \approx 0 \tag{2.5d}$$
$$\Omega_5 = [2\sigma_k\Pi_k + 2e\ A_0\ \sigma_k] \approx 0 \tag{2.5e}$$

where the first two constraints Ω_1 and Ω_2 are primary constraints and the last three Ω_3, Ω_4, and Ω_5 are secondary. Also, Π_k, p_λ, Π_0, and E are the momenta canonically conjugate respectively to σ_k, λ, A_0, and A_1. The matrix of the Poisson brackets of the constraints Ω_i namely $M_{\alpha\beta}\,(z, z') := \{\Omega_\alpha(z), \Omega_\beta(z')\}_p$ is then calculated. The inverse of the matrix $M_{\alpha\beta}$ does not exist and therefore the matrix is singular implying that the set of constraints Ω_i is first-class and that the theory described by $\mathscr{L}$ is a GI theory (Mitra and Rajaraman, 1990a,b). In fact, the action of theory is seen to be invariant under the local vector gauge transformation (LVGT):

$$\delta\sigma_k = e\beta(x, t), \qquad \delta A_1 = \beta'(x, t), \qquad \delta A_0 = \mathring{\beta}(x, t) \tag{2.6a}$$
$$\delta\lambda = -\mathring{\beta}(x, t), \qquad \delta\Pi_k = \delta E = \delta\Pi_0 = \delta p_\lambda = 0 \tag{2.6b}$$

$$\delta u = \partial_0 \partial_0 \beta(x,t), \qquad \delta v = -\partial_0 \partial_0 \beta(x,t) \tag{2.6c}$$

$$\delta \Pi_u = 0, \qquad \delta \Pi_v = 0 \tag{2.6d}$$

where $\beta \equiv \beta(x,t)$ is an arbitrary function of its arguments. The generator of the LVGT is the charge operator of the theory

$$J^0 = \int j^0 dx = \int dx[e\beta(\mathring{\sigma}_k - eA_0) + \beta'(\mathring{A}_1 - A_0')] \tag{2.7}$$

and the current operator of the theory is

$$J^1 = \int j^1 dx = \int dx[e\beta(-\sigma_k' + eA_1) - \mathring{\beta}(\mathring{A}_1 - A_0')]. \tag{2.8}$$

The divergence of the vector–current density, namely $\partial_\mu j^\mu$ is therefore seen to vanish under the gauge constraint $\lambda_k \approx 0$. This implies that the theory possesses at the classical level, a local vector gauge symmetry (LVGS) under the gauge constraint $\lambda \approx 0$, which is, in fact, equivalent to the temporal or time-axial kind of a gauge for the coordinate λ.

The nonvanishing equal-time commutators of the theory, for example, under the gauge $\mathscr{G} = \lambda = 0$ are obtained as (Kulshreshtha, 2001):

$$[A_0(x), \Pi_0(y)] = i\left[\frac{-1}{e^2}\right]\delta(x-y) \tag{2.9a}$$

$$[A_1(x), \Pi_k(y)] = \left[\frac{-i}{e}\right]\delta'(x-y) \tag{2.9b}$$

$$[A_1(x), E(y)] = i\delta(x-y) \tag{2.9c}$$

$$[A_0(x), A_1(y)] = \left[\frac{i}{e^2}\right]\delta'(x-y) \tag{2.9d}$$

$$[\Pi_0(x), \Pi_k(y)] = i\left[\frac{-1}{e}\right]\delta(x-y) \tag{2.9e}$$

3. THE GNLSM IN THE LF FRAME

To study the theory in the LF frame, that is on the hyperplanes $x^+ = (x^0 + x^1)/\sqrt{2}$ = constant, one defines the coordinates $x^\pm := [(x^0 \pm x^1)/\sqrt{2}]$, and then writes all the quantities involved in the action in terms of $x^\pm$ instead of x^0 and x^1 (Dirac, 1949). The action of the theory in the LF frame thus reads

$$S = \int \mathscr{L}\, dx^+\, dx^- \tag{3.1a}$$

$$\mathscr{L} = \bigg[(\partial_+\sigma_k)(\partial_-\sigma_k) + \lambda\big(\sigma_k^2 - 1\big) + \frac{1}{2}(\partial_+ A^+ - \partial_- A^-)^2$$
$$- e[A^-(\partial_-\sigma_k) + A^+(\partial_+\sigma_k)] + e^2 A^+ A^-\bigg] \tag{3.1b}$$

$$A^{\mp} = A_{\pm} = (A_0 \pm A_1)/\sqrt{2}; \qquad \partial_\pm\sigma_k = (\mathring{\sigma}_k \pm \sigma_k')/\sqrt{2}. \tag{3.1c}$$

As shown previously, in (3.1b), the first term corresponds to a massless boson (which is equivalent to a massless fermion), the second term is the usual term involving the nonlinear constraint $[(\sigma_k^2 - 1) \approx 0]$ and the auxiliary field λ, the third term is the kinetic energy term of the electromagnetic vector-gauge field $A_\mu(x, t)$, the fourth term represents the coupling of the sigma field to the electromagnetic field, and the last term is the mass term for the vector gauge boson A_μ.

The Euler–Lagrange equations obtained from $\mathscr{L}$ (3.1) are

$$[2\partial_+\partial_-\,\sigma_k] = [e(\partial_+ A^+ + \partial_- A^-) + 2\lambda\,\sigma_k] \tag{3.2a}$$
$$[\partial_+(\partial_+ A^+ - \partial_- A^-)] = [e^2 A^- - e(\partial_+\sigma_k)] \tag{3.2b}$$
$$[\partial_-(\partial_+ A^+ - \partial_- A^-)] = [e(\partial_-\sigma_k) - e^2 A^+] \tag{3.2c}$$
$$\big[\sigma_k^2 - 1\big] = 0. \tag{3.2d}$$

3.1. The Hamiltonian Formulation

The light-cone canonical momenta for the above LC–GNLSM obtained from $\mathscr{L}$ are

$$\Pi_k := \frac{\partial\mathscr{L}}{\partial(\partial_+\sigma_k)} = [\partial_-\sigma_k - eA^+] \tag{3.3a}$$
$$p_\lambda := \frac{\partial\mathscr{L}}{\partial(\partial_+\lambda)} = 0 \tag{3.3b}$$
$$\Pi^+ := \frac{\partial\mathscr{L}}{\partial(\partial_+ A^-)} = 0 \tag{3.3c}$$
$$\Pi^- := \frac{\partial\mathscr{L}}{\partial(\partial_+ A^+)} = [\partial_+ A^+ - \partial_- A^-] \tag{3.3d}$$

here, Π^+, Π^-, Π_k, and p_λ are the momenta canonically conjugate respectively to A^-, A^+, σ_k, and λ. Also these equations imply that the theory possesses three primary constraints:

$$\chi_1 = \Pi^+ \approx 0 \tag{3.4a}$$
$$\chi_2 = p_\lambda \approx 0 \tag{3.4b}$$
$$\chi_3 = [\Pi_k - \partial_-\sigma_k + eA^+] \approx 0. \tag{3.4c}$$

The canonical Hamiltonian density corresponding to $\mathscr{L}$ is

$$\mathscr{H}_C = [\Pi_k(\partial_+\sigma_k) + \Pi^+(\partial_+A^-) + \Pi^-(\partial_+A^+) + p_\lambda(\partial_+\lambda) - \mathscr{L}]$$
$$= \left[\frac{1}{2}(\Pi^-)^2 + \Pi^-(\partial_-A^-) - eA^-(\partial_-\sigma_k) - \lambda\left(\sigma_k^2 - 1\right) - e^2A^+A^-\right]. \quad (3.5)$$

After including the primary constraints χ_1, χ_2, and χ_3 in the canonical Hamiltonian density $\mathscr{H}_C$ with the help of Lagrange multipliers u_1, u_2, and u_3, one can write the total Hamiltonian density $\mathscr{H}_T$ as

$$\mathscr{H}_T = \left[\frac{1}{2}(\Pi^-)^2 + \Pi^-(\partial_-A^-) - eA^-(\partial_-\sigma_k) - \lambda\left(\sigma_k^2 - 1\right) - e^2A^+A^- \right.$$
$$\left. + \Pi^+u_1 + p_\lambda u_2 + (\Pi_k - \partial_-\sigma_k + eA^+)u_3\right] \quad (3.6)$$

the Hamiltons equations obtained from the total Hamiltonian $H_T = \int \mathscr{H}_T\, dx^-$ are

$$\partial_+\sigma_k = \frac{\partial H_T}{\partial \Pi_k} = u_3 \quad (3.7a)$$

$$-\partial_+\Pi_k = \frac{\partial H_T}{\partial \sigma_k} = [e\partial_-A^- - 2\lambda\sigma_k + \partial_-u_3] \quad (3.7b)$$

$$\partial_+\lambda = \frac{\partial H_T}{\partial p_\lambda} = u_2 \quad (3.7c)$$

$$-\partial_+p_\lambda = \frac{\partial H_T}{\partial \lambda} = \left[-\left(\sigma_k^2 - 1\right)\right] \quad (3.7d)$$

$$\partial_+A^- = \frac{\partial H_T}{\partial \Pi^+} = u_1 \quad (3.7e)$$

$$-\partial_+\Pi^- = \frac{\partial H_T}{\partial A^+} = [-e^2A^- + eu_3] \quad (3.7f)$$

$$\partial_+A^+ = \frac{\partial H_T}{\partial \Pi^-} = [\Pi^- + \partial_-A^-] \quad (3.7g)$$

$$-\partial_+\Pi^+ = \frac{\partial H_T}{\partial A^-} = [-\partial_-\Pi^- - e\partial_-\sigma_k - e^2A^+] \quad (3.7h)$$

$$\partial_+u_1 = \frac{\partial H_T}{\partial \Pi u_1} = 0 \quad (3.7i)$$

$$-\partial_+\Pi u_1 = \frac{\partial H_T}{\partial u_1} = \Pi^+ \quad (3.7j)$$

$$\partial_+u_2 = \frac{\partial H_T}{\partial \Pi u_2} = 0 \quad (3.7k)$$

$$-\partial_{+}\Pi u_2 = \frac{\partial H_{\mathrm{T}}}{\partial u_2} = p_{\lambda} \tag{3.7l}$$

$$\partial_{+}u_3 = \frac{\partial H_{\mathrm{T}}}{\partial \Pi u_3} = 0 \tag{3.7m}$$

$$-\partial_{+}\Pi u_3 = \frac{\partial H_{\mathrm{T}}}{\partial u_3} = [\Pi_k - \partial_{-}\sigma_k + eA^{+}]. \tag{3.7n}$$

These are the equations of motion that preserve the constraints of the theory χ_1, χ_2, and χ_3 in the course of time. For the equal light-cone-time ($x^+ = y^+$), Poisson bracket $\{,\}_{\mathrm{p}}$ of two functions A and B, we choose the convention

$$\{A(x), B(y)\}_{\mathrm{p}} := \int dz^{-} \sum_{\alpha} \left[\frac{\partial A(x)}{\partial q_{\alpha}(z)} \frac{\partial B(y)}{\partial p_{\alpha}(z)} - \frac{\partial A(x)}{\partial p_{\alpha}(z)} \frac{\partial B(y)}{\partial q_{\alpha}(z)} \right] \tag{3.8}$$

demanding that primary constraint χ_1 be preserved in the course of time, we obtain the secondary constraint

$$\chi_4 := \{\chi_1, \mathscr{H}_{\mathrm{T}}\}_{\mathrm{p}} = [\partial_{-}\Pi^{-} + e(\partial_{-}\sigma_k) + e^2 A^{+}] \approx 0. \tag{3.9}$$

Similarly, demanding the preservation of χ_2 in the course of time leads to the secondary constraint:

$$\chi_5 := \{\chi_2, \mathscr{H}_{\mathrm{T}}\}_{\mathrm{p}} = \left[\sigma_k^2 - 1\right] \approx 0. \tag{3.10}$$

Now the preservation of χ_3, χ_4, and χ_5 for all time does not give rise to any further constraints. The theory is thus seen to possess a set of five constraints $\chi_i (i = 1, \ldots, 5)$:

$$\chi_1 = \Pi^{+} \approx 0 \tag{3.11a}$$

$$\chi_2 = p_{\lambda} \approx 0 \tag{3.11b}$$

$$\chi_3 = [\Pi_k - \partial_{-}\sigma_k + eA^{+}] \approx 0 \tag{3.11c}$$

$$\chi_4 = [\partial_{-}\Pi^{-} + e(\partial_{-}\sigma_k) + e^2 A^{+}] \approx 0 \tag{3.11d}$$

$$\chi_5 = \left[\sigma_k^2 - 1\right] \approx 0. \tag{3.11e}$$

The matrix of the Poisson brackets of the constraints χ_i, namely $S_{\alpha\beta}(w^-, z^-) := \{\chi_{\alpha}(w^-), \chi_{\beta}(z^-)\}_{\mathrm{p}}$ is then calculated. The nonvanishing matrix elements of the matrix $S_{\alpha\beta}(w^-, z^-)$ (with the arguments of the field variables being suppressed) are

$$S_{33} = [-2\partial_{-}\delta(w^- - z^-)] \tag{3.12a}$$

$$S_{35} = -S_{53} = [-2\,\sigma_k\,\delta(w^- - z^-)] \tag{3.12b}$$

$$S_{44} = [-2e^2\partial_{-}\delta(w^- - z^-)]. \tag{3.12c}$$

The inverse of the matrix $S_{\alpha\beta}$ does not exist and therefore the matrix is singular implying that the set of constraints χ_i is first-class and that the theory is a GI theory (Mitra and Rajaraman, 1990a,b). In fact, the action of theory is seen to be invariant under the local vector gauge transformation (LVGT):

$$\delta\sigma_k = e\beta, \quad \delta A^+ = \partial_-\beta, \quad \delta A^- = \partial_+\beta, \quad \delta\lambda = -\partial_+\beta \tag{3.13a}$$

$$\delta\Pi_k = \delta\Pi^+ = \delta\Pi^- = \delta p_\lambda = \delta\Pi u_1 = \delta\Pi u_2 = \delta\Pi u_3 = 0 \tag{3.13b}$$

$$\delta u_1 = \partial_+\partial_+\beta, \quad \delta u_2 = -\partial_+\partial_+\beta, \quad \delta u_3 = e\partial_+\beta \tag{3.13c}$$

where $\beta \equiv \beta(x^-, x^+)$ is an arbitrary function of its arguments.

The generator of the above LVGT is the charge operator of the theory

$$J^+ = \int j^+ dx^- = \int dx^- [e\beta(\partial_-\sigma_k) - e^2\beta A^+ + (\partial_-\beta)(\partial_+ A^+ - \partial_- A^-)] \tag{3.14}$$

and the current operator of the theory is

$$J^- = \int j^- dx^- = \int dx^- [e\beta(\partial_+\sigma_k) - e^2\beta A^- - (\partial_+\beta)(\partial_+ A^+ - \partial_- A^-)]. \tag{3.15}$$

The divergence of the vector–current density, namely $\partial_\mu j^\mu (=\partial_+ j^+ + \partial_- j^-)$ is therefore seen to vanish under the gauge constraint $\lambda \approx 0$. This implies that the theory possesses at the classical level, a local vector gauge symmetry under the gauge $\lambda \approx 0$ that is equivalent to the temporal or time-axial kind of a gauge for the coordinate λ.

We now proceed to quantize the theory under the gauge

$$\mathscr{G}_1 = A^- = 0 \tag{3.16a}$$

$$\mathscr{G}_2 = \lambda = 0. \tag{3.16b}$$

Under this gauge, the total set of constraints of the theory becomes

$$\psi_1 = \chi_1 = \Pi^+ \approx 0 \tag{3.17a}$$

$$\psi_2 = \chi_2 = p_\lambda \approx 0 \tag{3.17b}$$

$$\psi_3 = \chi_3 = [\Pi_k - \partial_-\sigma_k + eA^+] \approx 0 \tag{3.17c}$$

$$\psi_4 = \chi_4 = [\partial_-\Pi^- + e(\partial_-\sigma_k) + e^2 A^+] \approx 0 \tag{3.17d}$$

$$\psi_5 = \chi_5 = \left[\sigma_k^2 - 1\right] \approx 0 \tag{3.17e}$$

$$\psi_6 = \mathscr{G}_1 = A^- = 0 \tag{3.17f}$$

$$\psi_7 = \mathscr{G}_2 = \lambda = 0. \tag{3.17g}$$

The matrix of the Poisson brackets of the constraints ψ_i, namely $T_{\alpha\beta}(w, z) := \{\psi_\alpha(w), \psi_\beta(z)\}_{\mathrm{p}}$ is then calculated. The nonvanishing matrix elements of the matrix $T_{\alpha\beta}(w, z)$ (with the arguments of the field variables being suppressed again) are

$$T_{16} = -T_{61} = -\delta(w^- - z^-) \tag{3.18a}$$

$$T_{27} = -T_{72} = -\delta(w^- - z^-) \tag{3.18b}$$

$$T_{33} = -2\partial_-\delta(w^- - z^-) \tag{3.18c}$$

$$T_{35} = -T_{53} = -2\sigma_k\delta(w^- - z^-) \tag{3.18d}$$

$$T_{44} = -2e^2\partial_-\delta(w^- - z^-). \tag{3.18e}$$

The inverse of the matrix $T_{\alpha\beta}$ exists and the matrix is nonsingular. The nonvanishing elements of the inverse of the matrix $T_{\alpha\beta}$ (i.e., the elements of the matrix $(T^{-1})_{\alpha\beta}$ (with the arguments of the field variables being suppressed once again) are

$$(T^{-1})_{16} = -(T^{-1})_{61} = \delta(w^- - z^-) \tag{3.19a}$$

$$(T^{-1})_{27} = -(T^{-1})_{72} = \delta(w^- - z^-) \tag{3.19b}$$

$$(T^{-1})_{35} = -(T^{-1})_{53} = \left[\frac{1}{2\sigma_k}\right]\delta(w^- - z^-) \tag{3.19c}$$

$$(T^{-1})_{44} = \left[\frac{-1}{2e^2}\right]\frac{1}{2} \in (w^- - z^-) \tag{3.19d}$$

$$(T^{-1})_{55} = \left[\frac{-1}{2\sigma_k^2}\right]\partial_-\delta(w^- - z^-) \tag{3.19e}$$

with $\in (w^- - z^-)$ being the step function, and

$$\int dz^- T(x^-, z^-)T^{-1}(z^-, y^-) = 1_{7x7}\,\delta(x^- - y^-). \tag{3.20}$$

The Dirac bracket $\{,\}_{\mathrm{D}}$ of two functions A and B is defined as (Dirac, 1950, 1964)

$$\begin{aligned}\{A, B\}_{\mathrm{D}} := \{A, B\}_{\mathrm{p}} - \int\int dw^- dz^- \\ \times \sum_{\alpha,\beta} \left[\{A, \Gamma_\alpha\,(w)\}_{\mathrm{p}}\left[\Delta^{-1}_{\alpha\beta}(w, z)\right]\{\Gamma_\beta(z), B\}_{\mathrm{p}}\right]\end{aligned} \tag{3.21}$$

where Γ_i are the constraints of the theory and $\Delta_{\alpha\beta}(w, z)[:= \{\Gamma_\alpha(w), \Gamma_\beta(z)\}_{\mathrm{p}}]$ is the matrix of the Poisson brackets of the constraints Γ_i. The transition to quantum theory is made by the replacement of the Dirac brackets by the operator commutation relations according to

$$\{A, B\}_{\mathrm{D}} \to (-i)[A, B]; \quad i = \sqrt{-1}. \tag{3.22}$$

Finally, the nonvanishing equal light-cone time commutators of the theory under the gauge (3.16) are obtained as

$$[A^+(x), \Pi^-(y)] = \frac{3}{2} i\delta(x^- - y^-) \tag{3.23a}$$

$$[A^+(x), \Pi_k(y)] = \frac{1}{2} i\partial_-\delta(x^- - y^-) \tag{3.23b}$$

$$[A^+(x), A^+(y)] = \left[\frac{-1}{2e^2}\right] i\partial_-\delta(x^- - y^-) \tag{3.23c}$$

$$[\Pi^-(x), \Pi^-(y)] = \left[\frac{-e^2}{4}\right] i \in (x^- - y^-) \tag{3.23d}$$

$$[\Pi^-(x), \Pi_k(y)] = \frac{1}{2} ie\delta(x^- - y^-) \tag{3.23e}$$

$$[\Pi_k(x), \Pi_k(y)] = \left[\frac{-1}{2}\right] i\partial_-\delta(x^- - y^-). \tag{3.23f}$$

Also, for later use (in the next section), to consider the BRST formulation of our GI theory, we convert the total Hamiltonian density $\mathscr{H}_\mathrm{T}$ into the first-order Lagrangian density:

$$\begin{aligned}\mathscr{L}_\mathrm{IO} := {}&[\Pi^+(\partial_+A^-) + \Pi^-(\partial_+A^+) + \Pi_k(\partial_+\sigma_k) + p_\lambda(\partial_+\lambda) + \Pi u_1(\partial_+u_1)\\ &+ \Pi u_2(\partial_+u_2) + \Pi u_3(\partial_+u_3) - \mathscr{H}_\mathrm{T}]\end{aligned} \tag{3.24a}$$

$$\begin{aligned}\mathscr{L}_\mathrm{IO} := {}&\Bigg[\Pi^-(\partial_+A^+) - \frac{1}{2}(\Pi^-)^2 - \Pi^-(\partial_-A^-) + eA^-(\partial_-\sigma_k) + \lambda\big(\sigma_k^2 - 1\big)\\ &+ e^2A^+A^- + (\partial_-\sigma_k - eA^+)(\partial_+\sigma_k) + \Pi u_1(\partial_+u_1) + \Pi u_2(\partial_+u_2)\\ &+ \Pi u_3(\partial_+u_3)\Bigg].\end{aligned} \tag{3.24b}$$

In the previous equation the terms $\Pi^+(\partial_+A^- - u_1)$, $p_\lambda(\partial_+\lambda - u_2)$ and $\Pi_k(\partial_+ \sigma_k - u_3)$ drop out in view of the Hamiltons equations of the theory.

4. THE BRST FORMULATION

We now rewrite our GNLSM, that is GI, as a quantum system that possesses the generalized gauge invariance called BRST symmetry. For this, we first enlarge the Hilbert space of our gauge-invariant GNLSM and replace the notion of gauge transformation, which shifts operators by c-number functions, by a BRST transformation, which mixes operators with Bose and Fermi statistics. We then introduce new anticommuting variables c and $\bar{c}$ (Grassmann numbers on the classical level, operators in the quantized theory) and a commuting variable

b (called the Nakamishi–Lautrup field) such that (Becchi *et al.*, 1974; Henneaux and Teitelboim, 1992; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshta *et al.*, 1993b, 1994a,b,c, 1995, 1999; Mitra and Rajaraman, 1990a,b; Nemeschansky *et al.*, 1988; Tyutin, 1975)

$$\hat{\delta}\sigma_k = ec; \quad \hat{\delta}A^+ = \partial_- c; \quad \hat{\delta}A^- = \partial_+ c; \quad \hat{\delta}\lambda = -\partial_+ c \tag{4.1a}$$

$$\hat{\delta}\Pi_k = \hat{\delta}\Pi^+ = \hat{\delta}\Pi^- = \hat{\delta}p_\lambda = 0; \quad \hat{\delta}u_1 = \partial_+\partial_+ c; \quad \hat{\delta}u_2 = -\partial_+\partial_+ c \tag{4.1b}$$

$$\hat{\delta}\Pi u_1 = \hat{\delta}\Pi u_2 = \hat{\delta}\Pi u_3 = 0; \qquad \hat{\delta}u_3 = e\partial_+ c \tag{4.1c}$$

$$\hat{\delta}c = 0; \qquad \hat{\delta}\bar{c} = b; \qquad \hat{\delta}b = 0 \tag{4.1d}$$

with the property $\hat{\delta}^2 = 0$. We now define a BRST-invariant function of the dynamical variables to be a function $f(\Pi_k, p_\lambda, \Pi^+, \Pi^-, \Pi_{u_1}, \Pi_{u_2}, \Pi_{u_3}, p_b, \Pi_c, \Pi_{\bar{c}}, \sigma_k, \lambda, A^+, A^-, u_1, u_2, u_3, b, c, \bar{c})$ such that $\hat{\delta} f = 0$.

4.1. Gauge Fixing in the BRST Formalism

Performing gauge-fixing in the BRST formalism implies adding to the first-order Lagrangian density $\mathscr{L}_{\text{IO}}$ a trivial BRST-invariant function (Becchi *et al.*, 1974; Henneaux and Teitelboim, 1992; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshta *et al.*, 1993b, 1994a,b,c, 1995, 1999; Mitra and Rajaraman, 1990a,b; Nemeschansky *et al.*, 1988; Tyutin, 1975). We thus write

$$\begin{aligned}\mathscr{L}_{\text{BRST}} = \Bigg[&\Pi^-(\partial_+ A^+) - \frac{1}{2}(\Pi^-)^2 - \Pi^-(\partial_- A^-) + eA^-(\partial_-\sigma_k) + \lambda\left(\sigma_k^2 - 1\right) \\ &+ e^2 A^+ A^- + (\partial_-\sigma_k - eA^+)(\partial_+\sigma_k) + \Pi_{u_1}(\partial_+ u_1) + \Pi_{u_2}(\partial_+ u_2) \\ &+ \Pi_{u_3}(\partial_+ u_3) + \hat{\delta}\left(\bar{c}\left(\partial_+ A^- + \frac{1}{e}\sigma_k + \frac{1}{2}b\right)\right)\Bigg].\end{aligned} \tag{4.2}$$

The last term in the previous equation is the extra BRST-invariant gauge-fixing term. After one integration by parts, this equation could now be written as

$$\begin{aligned}\mathscr{L}_{\text{BRST}} = \Bigg[&\Pi^-(\partial_+ A^+) - \frac{1}{2}(\Pi^-)^2 - \Pi^-(\partial_- A^-) + eA^-(\partial_-\sigma_k) + \lambda\left(\sigma_k^2 - 1\right) \\ &+ e^2 A^+ A^- + (\partial_-\sigma_k - eA^+)(\partial_+\sigma_k) + \Pi_{u_1}(\partial_+ u_1) + \Pi_{u_2}(\partial_+ u_2) \\ &+ \Pi_{u_3}(\partial_+ u_3) + b\left(\partial_+ A^- + \frac{1}{e}\sigma_k\right) + \frac{1}{2}b^2 + (\partial_+\bar{c})(\partial_+ c) - \bar{c}c\Bigg].\end{aligned} \tag{4.3}$$

Proceeding classically, the Euler–Lagrange equation for b reads

$$-b = \left(\partial_+ A^- + \frac{1}{e}\,\sigma_k\right) \tag{4.4}$$

the requirement $\hat{\delta} b = 0$ then implies

$$-\hat{\delta} b = \left(\hat{\delta} \partial_+ A^- + \frac{1}{e} \, \hat{\delta} \sigma_k \right) \tag{4.5}$$

which in turn implies

$$-\partial_+ \, \partial_+ \, c = c. \tag{4.6}$$

This equation is also an Euler–Lagrange equation obtained by the variation of $\mathscr{L}_{\text{BRST}}$ with respect to $\overline{c}$. In introducing momenta, one has to be careful in defining those for the fermionic variables. We thus define the bosonic momenta in the usual manner so that

$$\Pi^+ := \frac{\partial}{\partial(\partial_+ A^-)} \mathscr{L}_{\text{BRST}} = b \tag{4.7}$$

but for the fermionic momenta with directional derivatives we set

$$\Pi_c = \mathscr{L}_{\text{BRST}} \frac{\overleftarrow{\partial}}{\partial(\partial_+ c)} = (\partial_+ \overline{c}); \qquad \Pi_{\overline{c}} = \frac{\overrightarrow{\partial}}{\partial(\partial_+ \overline{c})} \mathscr{L}_{\text{BRST}} = (\partial_+ c) \tag{4.8}$$

implying that the variable canonically conjugate to c is $(\partial_+ \overline{c})$ and the variable conjugate to $\overline{c}$ is $(\partial_+ c)$. For writing the Hamiltonian density from the Lagrangian density in the usual manner we remember that the former has to be Hermitian so that

$$\begin{aligned} \mathscr{H}_{\text{BRST}} = {} & [\Pi_k(\partial_+ \sigma_k) + p_\lambda(\partial_+ \lambda) + \Pi^+(\partial_+ A^-) + \Pi^-(\partial_+ A^+) + \Pi_{u_1}(\partial_+ u_1) \\ & + \Pi_{u_2}(\partial_+ u_2) + \Pi_{u_3}(\partial_+ u_3) + \Pi_c(\partial_+ c) + (\partial_+ \overline{c}) \Pi_{\overline{c}} - \mathscr{L}_{\text{BRST}}] \end{aligned} \tag{4.9a}$$

$$\begin{aligned} \mathscr{H}_{\text{BRST}} = {} & \left[\frac{1}{2} (\Pi^-)^2 + \Pi^-(\partial_- A^-) - e A^-(\partial_- \sigma_k) - e^2 A^+ A^- - \frac{1}{e} \sigma_k \Pi^+ \right. \\ & - \frac{1}{2} (\Pi^+)^2 + p_\lambda(\partial_+ \lambda) + (\Pi_k - \partial_- \sigma_k + e A^+)(\partial_+ \sigma_k) \\ & \left. - \lambda \left(\sigma_k^2 - 1 \right) + \Pi_c \Pi_{\overline{c}} + \overline{c} c \right] \end{aligned} \tag{4.9b}$$

We can check the consistency of (4.8) and (4.9) by looking at Hamiltons equations for the fermionic variables, that is,

$$\partial_+ c = \frac{\overrightarrow{\partial}}{\partial \Pi_c} \mathscr{H}_{\text{BRST}}, \qquad \partial_+ \overline{c} = \mathscr{H}_{\text{BRST}} \frac{\overleftarrow{\partial}}{\partial \Pi_{\overline{c}}}. \tag{4.10}$$

Thus we see that

$$\partial_+ c = \frac{\vec{c}}{\partial \Pi_c} \mathscr{H}_{\text{BRST}} = \Pi_{\overline{c}}; \qquad \partial_+ \overline{c} = \mathscr{H}_{\text{BRST}} \frac{\overleftarrow{\partial}}{\partial \Pi_{\overline{c}}} = \Pi_c \tag{4.11}$$

is in agreement with (4.8). For the operators c, $\bar{c}$, $\partial_+ c$, and $\partial_+ \bar{c}$, one needs to satisfy the anticommutation relations of $\partial_+ c$ with $\bar{c}$ or of $\partial_+ \bar{c}$ with c, but not of c, with $\bar{c}$. In general, c and $\bar{c}$ are independent canonical variables and one assumes that

$$\{\Pi_c, \Pi_{\bar{c}}\} = \{\bar{c}, c\} = 0; \quad \partial_+\{\bar{c}, c\} = 0 \tag{4.12a}$$

$$\{\partial_+\bar{c}, c\} = (-1)\{\partial_+ c, \bar{c}\} \tag{4.12b}$$

where {,} means an anticommutator. We thus see that the anticommutators in (4.12b) are nontrivial and need to be fixed. To fix these, we demand that c satisfies the Heisenberg equation (Becchi *et al.*, 1974; Henneaux and Teitelboim, 1992; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshta *et al.*, 1993b, 1994a,b,c, 1995, 1999; Mitra and Rajaraman, 1990a,b; Nemeschansky *et al.*, 1988; Tyutin, 1975):

$$[c, \mathscr{H}_{\text{BRST}}] = i\,\partial_+ c \tag{4.13}$$

and using the property $c^2 = \bar{c}^2 = 0$, one obtains

$$[c, \mathscr{H}_{\text{BRST}}] = \{\partial_+\bar{c}, c\}\partial_+ c. \tag{4.14}$$

Equations (4.12)–(4.14) then imply

$$\{\partial_+\bar{c}, c\} = (-1)\{\partial_+ c, \bar{c}\} = i \tag{4.15}$$

here the minus sign in this equation is nontrivial and implies the existence of states with negative norm in the space of state vectors of the theory (Becchi *et al.*, 1974; Henneaux and Teitelboim, 1992; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 1998; Kulshreshta *et al.*, 1993b, 1994a,b,c, 1995, 1999; Mitra and Rajaraman, 1990a,b; Nemeschansky *et al.*, 1988; Tyutin, 1975).

4.2. The BRST Charge Operator

The BRST charge operator Q is the generator of the BRST transformations (4.1). It is nilpotent and satisfies $Q^2 = 0$. It mixes operators that satisfy Bose and Fermi statistics. According to its conventional definition, its commutators with Bose operators and its anticommutators with Fermi operators for the present theory satisfy

$$[\sigma_k, Q] = \partial_+ c; \qquad [\Pi_k, Q] = [2c\sigma_k + \partial_-\partial_+ c] \tag{4.16a}$$

$$[\lambda, Q] = \partial_+ c; \qquad [A^+, Q] = -\partial_- c; \quad [A^-, Q] = \partial_+ c \tag{4.16b}$$

$$[\Pi^-, Q] = [e^2 c - e\partial_+ c] \tag{4.16c}$$

$$\{\bar{c}, Q\} = [\partial_-\sigma_k - eA^+ - \Pi^+ - p_\lambda - \Pi_k] \tag{4.16d}$$

$$\{\partial_+\bar{c}, Q\} = (-1)\left[\partial_-\Pi^- + e\partial_-\sigma_k + e^2 A^+ + \sigma_k^2 - 1\right]. \tag{4.16e}$$

All other commutators and anticommutators involving Q vanish. In view of (4.16), the BRST charge operator of the present theory can be written as

$$Q = \int dx^- \big[ic\big[\partial_- \Pi^- + e\partial_- \sigma_k + e^2 A^+ + \sigma_k^2 - 1\big] \\ - i(\partial_+ c)[\Pi^+ + p_\lambda + \Pi_k - \partial_- \sigma_k + eA^+]\big]. \tag{4.17}$$

This equation implies that the set of states satisfying the conditions

$$\Pi^+ |\psi\rangle = 0 \tag{4.18a}$$

$$p_\lambda |\psi\rangle = 0 \tag{4.18b}$$

$$[\Pi_k - \partial_- \sigma_k + eA^+] |\psi\rangle = 0 \tag{4.18c}$$

$$[\partial_- \Pi^- + e\partial_- \sigma_k + e^2 A^+] |\psi\rangle = 0 \tag{4.18d}$$

$$\big[\sigma_k^2 - 1\big] |\psi\rangle = 0 \tag{4.18e}$$

belongs to the dynamically stable subspace of states $|\psi\rangle$ satisfying $Q|\psi\rangle = 0$, that is, it belongs to the set of BRST-invariant states.

To understand the condition needed for recovering the physical states of the theory, we rewrite the operators c and $\bar{c}$ in terms of fermionic annihilation and creation operators. For this purpose we consider (4.6). The solution of Eq. (4.6) gives the Heisenberg operator $c(t)$ (and correspondingly $\bar{c}(t)$) as ($t \equiv x^+$):

$$c(t) = e^{it} B + e^{-it} D; \qquad \bar{c}(t) = e^{-it} B^\dagger + e^{it} D^\dagger \tag{4.19}$$

which at time $t = 0$ imply

$$c \equiv c(0) = B + D; \qquad \bar{c} \equiv \bar{c}(0) = B^\dagger + D^\dagger \tag{4.20a}$$

$$\partial_+ c \equiv \partial_+ c(0) = i(B - D); \qquad \partial_+ \bar{c} \equiv \partial_+ \bar{c}(0) = -i(B^\dagger - D^\dagger). \tag{4.20b}$$

By imposing the conditions

$$c^2 = \bar{c}^2 = \{\bar{c}, c\} = \{\partial_+ \bar{c}, \partial_+ c\} = 0 \tag{4.21a}$$

$$\{\partial_+ \bar{c}, c\} = i = -\{\partial_+ c, \bar{c}\} \tag{4.21b}$$

we now obtain the equations

$$B^2 + \{B, D\} + D^2 = B^{\dagger 2} + \{B^\dagger, D^\dagger\} + D^{\dagger 2} = 0 \tag{4.22a}$$

$$\{B, B^\dagger\} + \{D, D^\dagger\} + \{B, D^\dagger\} + \{B^\dagger, D\} = 0 \tag{4.22b}$$

$$\{B, B^\dagger\} + \{D, D^\dagger\} - \{B, D^\dagger\} - \{B^\dagger, D\} = 0 \tag{4.22c}$$

$$\{B, B^\dagger\} - \{D, D^\dagger\} - \{B, D^\dagger\} + \{D, B^\dagger\} = -1 \tag{4.22d}$$

$$\{B, B^\dagger\} - \{D, D^\dagger\} + \{B, D^\dagger\} - \{D, B^\dagger\} = -1 \tag{4.22e}$$

with the solution

$$B^2 = D^2 = B^{\dagger 2} = D^{\dagger 2} = 0 \tag{4.23a}$$

$$\{B, D\} = \{B^\dagger, D\} = \{B, D^\dagger\} = \{B^\dagger, D^\dagger\} = 0 \tag{4.23b}$$

$$\{B^\dagger, B\} = -\frac{1}{2}; \qquad \{D^\dagger, D\} = \frac{1}{2}. \tag{4.23c}$$

We now let $|0\rangle$ denote the fermionic vacuum for which

$$B\,|0\rangle = D\,|0\rangle = 0 \tag{4.24}$$

defining $|0\rangle$ to have norm one, (4.23c) implies

$$\langle 0|\, BB^\dagger\, |0\rangle = -\frac{1}{2}; \qquad \langle 0|\, DD^\dagger\, |0\rangle = +\frac{1}{2} \tag{4.25}$$

so that

$$B^\dagger\,|0\rangle \neq 0; \qquad D^\dagger\,|0\rangle \neq 0. \tag{4.26}$$

The theory is thus seen to possess negative norm states in the fermionic sector. The existence of these negative norm states as free states of the fermionic part of $\mathscr{H}_{\mathrm{BRST}}$ is however, irrelevant to the existence of physical states in the orthogonal subspace of the Hilbert space.

In terms of annihilation and creation operators

$$\begin{aligned}\mathscr{H}_{\mathrm{BRST}} = \Bigg[&\frac{1}{2}(\Pi^-)^2 + \Pi^-(\partial_- A^-) - eA^-(\partial_-\sigma_k) - e^2 A^+ A^- - \frac{1}{e}\sigma_k \Pi^+ \\ &- \frac{1}{2}(\Pi^+)^2 + p_\lambda(\partial_+\lambda) + (\Pi_k - \partial_-\sigma_k + eA^+)(\partial_+\sigma_k) \\ &- \lambda\big(\sigma_k^2 - 1\big) + 2(B^\dagger B + D^\dagger D)\Bigg]\end{aligned} \tag{4.27}$$

and the BRST charge operator Q is

$$\begin{aligned}Q = \int dx^- \big[&iB\big[\big(\partial_-\Pi^- + e\partial_-\sigma_k + e^2 A^+ + \sigma_k^2 - 1\big) \\ &- i(\Pi^+ + p_\lambda + \Pi_k - \partial_-\sigma_k + eA^+)\big] + iD\big[\big(\partial_-\Pi^- + e\partial_-\sigma_k \\ &+ e^2 A^+ + \sigma_k^2 - 1\big) + i(\Pi^+ + p_\lambda + \Pi_k - \partial_-\sigma_k + eA^+)\big]\big].\end{aligned} \tag{4.28}$$

Now because $Q\,|\psi\rangle = 0$, the set of states annihilated by Q contains not only the set of states for which (4.18) hold but also additional states for which

$$B\,|\psi\rangle = D\,|\psi\rangle = 0 \tag{4.29a}$$

$$\Pi^+|\psi\rangle \neq 0 \tag{4.29b}$$

$$p_\lambda\,|\psi\rangle \neq 0 \tag{4.29c}$$

$$[\Pi_k - \partial_-\sigma_k + eA^+]\,|\psi\rangle \neq 0 \tag{4.29d}$$

$$[\partial_-\Pi^- + e\partial_-\sigma_k + e^2A^+]\,|\psi\rangle \neq 0 \tag{4.29e}$$

$$\left[\sigma_k^2 - 1\right]|\psi\rangle \neq 0. \tag{4.29f}$$

The Hamiltonian is also invariant under the anti-BRST transformation given by

$$\bar{\hat{\delta}}\sigma_k = -e\bar{c};\quad \bar{\hat{\delta}}A^+ = -\partial_-\bar{c};\quad \bar{\hat{\delta}}A^- = -\partial_+\bar{c};\quad \bar{\hat{\delta}}_\lambda = \partial_+\bar{c} \tag{4.30a}$$

$$\bar{\hat{\delta}}\Pi_k = \bar{\hat{\delta}}\Pi^+ = \bar{\hat{\delta}}\Pi^- = \bar{\hat{\delta}}p_\lambda = 0;\quad \bar{\hat{\delta}}u_1 = -\partial_+\partial_+\bar{c};\quad \bar{\hat{\delta}}u_2 = \partial_+\partial_+\bar{c} \tag{4.30b}$$

$$\bar{\hat{\delta}}\Pi_{u_1} = \bar{\hat{\delta}}\Pi_{u_2} = \bar{\hat{\delta}}\Pi_{u_3} = 0;\qquad \bar{\hat{\delta}}u_3 = -e\partial_+\bar{c} \tag{4.30c}$$

$$\bar{\hat{\delta}}\bar{c} = 0;\qquad \bar{\hat{\delta}}c = -b;\qquad \bar{\hat{\delta}}b = 0 \tag{4.30d}$$

with the generator or anti-BRST charge

$$\begin{aligned}\bar{Q} = \int dx^-\big[&-i\bar{c}\big[\partial_-\Pi^- + e\partial_-\sigma_k + e^2A^+ + \sigma_k^2 - 1\big]\\ &+ i(\partial_+\bar{c})[\Pi^+ + p_\lambda + \Pi_k - \partial_-\sigma_k + eA^+]\big]\end{aligned} \tag{4.31a}$$

$$\begin{aligned}\bar{Q} = \int dx^-\big[&-iB^\dagger\big[\big(\partial_-\Pi^- + e\partial_-\sigma_k + e^2A^+ + \sigma_k^2 - 1\big)\\ &+ i(\Pi^+ + p_\lambda + \Pi_k - \partial_-\sigma_k + eA^+)\big] - iD^\dagger\big[\big(\partial_-\Pi^- + e\partial_-\sigma_k\\ &+ e^2A^+ + \sigma_k^2 - 1\big) - i(\Pi^+ + p_\lambda + \Pi_k - \partial_-\sigma_k + eA^+)\big]\big]\end{aligned} \tag{4.31b}$$

we also have

$$\partial_+ Q = [Q, H_{\text{BRST}}] = 0 \tag{4.32a}$$

$$\partial_+ \bar{Q} = [\bar{Q}, H_{\text{BRST}}] = 0 \tag{4.32b}$$

with

$$H_{\text{BRST}} = \int dx\, \mathscr{H}_{\text{BRST}} \tag{4.32c}$$

and we further impose the dual conditions that both Q and $\bar{Q}$ annihilate physical states, implying that

$$Q\,|\psi\rangle = 0 \quad \text{and} \quad \bar{Q}\,|\psi\rangle = 0. \tag{4.33}$$

The states for which (4.18) hold satisfy both of these conditions and, in fact, are the only states satisfying both of these conditions, since although with (4.23)

$$2(B^\dagger B + D^\dagger D) = -2(BB^\dagger + DD^\dagger) \tag{4.34}$$

there are no states of this operator with $B^\dagger\,|0\rangle = 0$ and $D^\dagger\,|0\rangle = 0$ [cf. (4.26)], and hence no free eigenstates of the fermionic part of H_{BRST} which are annihilated by each of B, $B^\dagger$, D, $D^\dagger$. Thus the only states satisfying (4.33) are those satisfying the constraints (3.11).

Further, the states for which (4.18) hold satisfy both the conditions (4.33) and, in fact, are the only states satisfying both of these conditions because in view of (4.21), one cannot have simultaneously c, $\partial_+ c$ and $\overline{c}$, $\partial_+\overline{c}$, applied to $|\psi\rangle$ to give zero. Thus the only states satisfying (4.33) are those that satisfy the constraints of the theory (3.11) and they belong to the set of BRST-invariant and anti-BRST-invariant states.

Alternatively, one can understand the previous point in terms of fermionic annihilation and creation operators as follows. The condition $Q\,|\psi\rangle = 0$ implies that the set of states annihilated by Q contains not only the states for which (4.18) hold but also additional states for which (4.29) hold. However, $\overline{Q}\,|\psi\rangle = 0$ guarantees that the set of states annihilated by $\overline{Q}$ contains only the states for which (4.18) hold, simply because $B^\dagger\,|\psi\rangle \neq 0$ and $D^\dagger\,|\psi\rangle \neq 0$. Thus in this alternative way also, we see that the states satisfying $Q\,|\psi\rangle = \overline{Q}\,|\psi\rangle = 0$ (i.e., satisfying (4.33)) are only those states that satisfy the constraints of the theory and also that these states belong to the set of BRST invariant and anti-BRST-invariant states.

5. SUMMARY AND DISCUSSIONS

In this paper we have studied a GNLSM in the LF frame, that is, on the hyperplanes $x^+ = (x^0 + x^1)/\sqrt{2} =$ constant. The theory in the instant form (Kulshreshtha, 2001) is also seen to be GI possessing a set of five first-class constraints where two constraints are primary and three are secondary. The LF theory also possesses a set of five first-class constraints where three constraints are primary and two are secondary. The theory remains GI in both the cases as the matrix of the Poisson brackets of the constraints of the theory in both the cases (IF and LF) remains singular signifying that the sets of the (total number of) constraints in both the cases remains first-class. Also, in both the cases, there does not exist any problem with respect to the operator ordering as one encounters it in the case of usual (ungauged) NLSM.

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